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On error operators related to the arbitrary functions principle

Nicolas Bouleau

Ecole des Ponts, ParisTech, 28 rue des Saints Pères, 75007 Paris, France

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Abstract

The error on a real quantity Y due to the graduation of the measuring instrument may be asymptotically represented, when the graduation is regular and fines down, by a Dirichlet form on $\mathbb{R}$ whose square field operator does not depend on the probability law of Y as soon as this law possesses a continuous density. This feature is related to the "arbitrary functions principle" (Poincaré, Hopf). We give extensions of this property to $\mathbb{R}^d$ and to the Wiener space for some approximations of the Brownian motion. This gives new approximations of the Ornstein–Uhlenbeck gradient. These results apply to the discretization of some stochastic differential equations encountered in mechanics.

Keywords: Arbitrary functions; Dirichlet forms; Euler scheme; Girsanov theorem; Mechanical system; Rajchman measure; Square field operator; Stable convergence; Stochastic differential equation

0. Introduction

The approximation of a random variable Y by another one Y_n yields most often a Dirichlet form. The framework is general, cf. Bouleau [3] whose results are recalled in Section 1.1.

Usually, when this Dirichlet form exists and does not vanish, the conditional law of Y_n given $Y = y$ is not reduced to a Dirac mass, and the variance of this conditional law yields the square field operator Γ. On the other hand when the approximation is deterministic, i.e. when Y_n is a function of Y, say $Y_n = \eta_n(Y)$, then *most often* the symmetric bias operator $\widetilde{A}$ and the Dirichlet form vanish, cf. Bouleau [3, Examples 2.1–2.9 and Remark 5].

E-mail address: bouleau@enpc.fr.

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Nevertheless, there are cases where the conditional law of Y_n given Y is a Dirac mass, i.e. Y_n is a deterministic function of Y, and where the approximation of Y by Y_n yields even so a non-zero Dirichlet form on $L^2(\mathbb{P}_Y)$.

This phenomenon is interesting, insofar as randomness (here the Dirichlet form) is generated by a deterministic device. In its simplest form, the phenomenon appears precisely when a quantity is measured by a graduated instrument to the nearest graduation when looking at the asymptotic limits as the graduation fines down.

The first part of this article is devoted to functional analytical tools that we need afterwards. We first recall the properties of the bias operators and the Dirichlet form associated with an approximation. Next we prove a Girsanov-type theorem for Dirichlet forms which has its own interest, i.e. an answer to the question of an absolutely continuous change of measure for Dirichlet forms. At last we recall some simple properties of Rajchman measures.

The second part is devoted to the case of a real or finite-dimensional quantity measured with equidistant graduations. The mathematical argument here is basically *the arbitrary functions method* about which we give a short historical comment.

Several infinite-dimensional extensions of the arbitrary functions principle are studied in the third part. The first one is about approximations of continuous martingales whose brackets are Rajchman measures. Then we consider the case of the Wiener space on which the preceding results may be improved and other asymptotic properties are obtained concerning the approximation of the Ornstein–Uhlenbeck gradient. Eventually we apply these results to the approximation of stochastic differential equations encountered in mechanics and solved by the Euler scheme.

1. Functional analytical tools

1.1. Approximation, Dirichlet forms and bias operators

Our study uses the theoretical framework concerning the bias operators and the Dirichlet form generated by an approximation proposed in Bouleau [3]. We recall here the definitions and main results for the convenience of the reader. Here, considered Dirichlet forms are always symmetric.

Let Y be a random variable defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in a measurable space $(E, \mathcal{F})$ and let Y_n be approximations also defined on $(\Omega, \mathcal{A}, \mathbb{P})$ with values in $(E, \mathcal{F})$. We consider an algebra $\mathcal{D}$ of bounded functions from E into $\mathbb{R}$ or $\mathbb{C}$ containing the constants and dense in $L^2(E, \mathcal{F}, \mathbb{P}_Y)$ and a sequence α_n of positive numbers. With $\mathcal{D}$ and (α_n) we consider the four following assumptions defining the four bias operators:

$$\text{(H1)}\qquad \begin{cases} \forall\varphi \in \mathcal{D},\ \text{there exists } \overline{A}[\varphi] \in L^2(E, \mathcal{F}, \mathbb{P}_Y) \quad \text{such that } \forall\chi \in \mathcal{D}, \\ \lim_{n\to\infty} \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))\chi(Y)] = \mathbb{E}_Y[\overline{A}[\varphi]\chi]. \end{cases}$$

$$\text{(H2)}\qquad \begin{cases} \forall\varphi \in \mathcal{D},\ \text{there exists } \underline{A}[\varphi] \in L^2(E, \mathcal{F}, \mathbb{P}_Y) \quad \text{such that } \forall\chi \in \mathcal{D}, \\ \lim_{n\to\infty} \alpha_n \mathbb{E}[(\varphi(Y) - \varphi(Y_n))\chi(Y_n)] = \mathbb{E}_Y[\underline{A}[\varphi]\chi]. \end{cases}$$

$$\text{(H3)}\qquad \begin{cases} \forall\varphi \in \mathcal{D},\ \text{there exists } \widetilde{A}[\varphi] \in L^2(E, \mathcal{F}, \mathbb{P}_Y) \quad \text{such that } \forall\chi \in \mathcal{D}, \\ \lim_{n\to\infty} \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y_n) - \chi(Y))] = -2\mathbb{E}_Y[\widetilde{A}[\varphi]\chi]. \end{cases}$$

$$\text{(H4)}\qquad \begin{cases} \forall\varphi \in \mathcal{D},\ \text{there exists } A\!\!\!|[\varphi] \in L^2(E, \mathcal{F}, \mathbb{P}_Y) \quad \text{such that } \forall\chi \in \mathcal{D}, \\ \lim_{n\to\infty} \alpha_n \mathbb{E}[(\varphi(Y_n) - \varphi(Y))(\chi(Y_n) + \chi(Y))] = 2\mathbb{E}_Y[A\!\!\!|[\varphi]\chi]. \end{cases}$$

We first note that as soon as two of hypotheses (H1)–(H4) are fulfilled (with the same algebra $\mathcal{D}$ and the same sequence α_n), the other two follow thanks to the relations

$$\widetilde{A} = \frac{\overline{A} + \underline{A}}{2}, \qquad \not{A} = \frac{\overline{A} - \underline{A}}{2}.$$

When defined, the operator $\overline{A}$ which considers the asymptotic error from the point of view of the limit model, will be called *the theoretical bias operator*.

The operator $\underline{A}$ which considers the asymptotic error from the point of view of the approximating model will be called *the practical bias operator*.

Because of the property

$$\langle \widetilde{A}[\varphi], \chi \rangle_{L^2(\mathbb{P}_Y)} = \langle \varphi, \widetilde{A}[\chi] \rangle_{L^2(\mathbb{P}_Y)}$$

the operator $\widetilde{A}$ will be called *the symmetric bias operator*.

The operator $\not{A}$ which is often (see Theorem 2) a first-order operator will be called *the singular bias operator*.

Theorem 1. *Under the hypothesis* (H3)*,*

(a) *the limit*

$$\widetilde{\mathcal{E}}[\varphi, \chi] = \lim_n \frac{\alpha_n}{2} \mathbb{E}\big[\big(\varphi(Y_n) - \varphi(Y)\big)\big(\chi(Y_n) - \chi(Y)\big)\big], \quad \varphi, \chi \in \mathcal{D}, \tag{1}$$

defines a closable positive bilinear form which smallest closed extension is denoted $(\mathcal{E}, \mathbb{D})$.

(b) $(\mathcal{E}, \mathbb{D})$ *is a Dirichlet form.*

(c) $(\mathcal{E}, \mathbb{D})$ *admits a square field operator* Γ *satisfying* $\forall \varphi, \chi \in \mathcal{D}$,

$$\Gamma[\varphi] = \widetilde{A}\big[\varphi^2\big] - 2\varphi \widetilde{A}[\varphi], \tag{2}$$

$$\mathbb{E}_Y\big[\Gamma[\varphi]\chi\big] = \lim_n \alpha_n \mathbb{E}\big[\big(\varphi(Y_n) - \varphi(Y)\big)^2\big(\chi(Y_n) + \chi(Y)\big)/2\big]. \tag{3}$$

(d) $(\mathcal{E}, \mathbb{D})$ *is local if and only if* $\forall \varphi \in \mathcal{D}$,

$$\lim_n \alpha_n \mathbb{E}\big[\big(\varphi(Y_n) - \varphi(Y)\big)^4\big] = 0; \tag{4}$$

this condition is equivalent to

$$\exists \lambda > 2 \quad \lim_n \alpha_n \mathbb{E}\big[\big|\varphi(Y_n) - \varphi(Y)\big|^\lambda\big] = 0.$$

(e) *If the form* $(\mathcal{E}, \mathbb{D})$ *is local, then the principle of asymptotic error calculus is valid on* $\widetilde{\mathcal{D}} = \{F(f_1, \ldots, f_p)\colon\ f_i \in \mathcal{D},\ F \in \mathcal{C}^1(\mathbb{R}^p, \mathbb{R})\}$ *i.e.*

$$\lim_n \alpha_n \mathbb{E}\big[\big(F\big(f_1(Y_n),\dots,f_p(Y_n)\big)-F\big(f_1(Y),\dots,f_p(Y)\big)^2\big]$$
$$=\mathbb{E}_Y\left[\sum_{i,j=1}^{p} F_i'(f_1,\dots,f_p)F_j'(f_1,\dots,f_p)\Gamma[f_i,f_j]\right].$$

An operator B from $\mathcal{D}$ into $L^2(\mathbb{P}_Y)$ will be said to be a *first-order operator* if it satisfies

$$B[\varphi\chi]=B[\varphi]\chi+\varphi B[\chi] \quad \forall\varphi,\chi\in\mathcal{D}.$$

Theorem 2. *Under* (H1)–(H4). *If there is a real number* $p\geqslant 1$ *such that*

$$\lim_n \alpha_n \mathbb{E}\big[\big(\varphi(Y_n)-\varphi(Y)\big)^2\big|\psi(Y_n)-\psi(Y)\big|^p\big]=0 \quad \forall\varphi,\psi\in\mathcal{D}$$

then $\not{A}$ *is first-order.*

In particular, if the Dirichlet form is local, by the (d) of Theorem 1, the operator $\not{A}$ is first-order.

1.2. Girsanov-type theorem for Dirichlet forms

An error structure is a probability space $(\Omega,\mathcal{A},\mathbb{P})$ equipped with a local Dirichlet form with domain $\mathbb{D}$ dense in $L^2(\Omega,\mathcal{A},\mathbb{P})$ admitting a square field operator Γ, see Bouleau [2]. We denote $\mathcal{D}A$ the domain of the associated generator.

Theorem 3. *Let* $(\Omega,\mathcal{A},\mathbb{P},\mathbb{D},\Gamma)$ *be an error structure. Let be* $f\in\mathbb{D}\cap L^\infty$ *such that* $f>0$, $\mathbb{E}f=1$. *We put* $\mathbb{P}_1=f.\mathbb{P}$.

(a) *The bilinear form* $\mathcal{E}_1$ *defined on* $\mathcal{D}A\cap L^\infty$ *by*

$$\mathcal{E}_1[u,v]=-\mathbb{E}\left[fvA[u]+\frac{1}{2}v\Gamma[u,f]\right] \tag{5}$$

is closable in $L^2(\mathbb{P}_1)$ *and satisfies for* $u,v\in\mathcal{D}A\cap L^\infty$,

$$\mathcal{E}_1[u,v]=-\langle A_1u,v\rangle=-\langle u,A_1v\rangle=\frac{1}{2}\mathbb{E}\big[f\Gamma[u,v]\big], \tag{6}$$

where $A_1[u]=A[u]+\frac{1}{2f}\Gamma[u,f]$.

(b) *Let* $(\mathbb{D}_1,\mathcal{E}_1)$ *be the smallest closed extension of* $(\mathcal{D}A\cap L^\infty,\mathcal{E}_1)$. *Then* $\mathbb{D}\subset\mathbb{D}_1$, $\mathcal{E}_1$ *is local and admits a square field operator* Γ_1, *and*

$$\Gamma_1=\Gamma \quad \text{on } \mathbb{D};$$

in addition, $\mathcal{D}A\subset\mathcal{D}A_1$ *and* $A_1[u]=A[u]+\frac{1}{2f}\Gamma[u,f]$ *for all* $u\in\mathcal{D}A$.

Proof. (1) First, using that the resolvent operators are bounded operators sending L^∞ into $\mathcal{D}A \cap L^\infty$, we see that $\mathcal{D}A \cap L^\infty$ is dense in $\mathbb{D}$ (equipped with the usual norm $(\|\cdot\|_{L^2}^2 + \mathcal{E}[\cdot])^{1/2}$), hence also dense in $L^2(\mathbb{P}_1)$.

(2) Using that $\mathbb{D} \cap L^\infty$ is an algebra, for $u, v \in \mathcal{D}A \cap L^\infty$ we have

$$\mathcal{E}_1[u,v] = -\mathbb{E}\left[f v A[u] + \frac{1}{2} v \Gamma[u,f]\right] = \frac{1}{2}\mathbb{E}\big[\Gamma[fv,u] - v\Gamma[u,f]\big] = \frac{1}{2}\mathbb{E}\big[f\Gamma[u,v]\big].$$

So, defining A_1 as in the statement, we have $\forall u, v \in \mathcal{D}A \cap L^\infty$,

$$\mathcal{E}_1[u,v] = -\mathbb{E}_1[vA_1u] = -\mathbb{E}_1[uA_1v].$$

The operator A_1 is therefore symmetric on $\mathcal{D}A \cap L^\infty$ under $\mathbb{P}_1$. Hence the form $\mathcal{E}_1$ defined on $\mathcal{D}A \cap L^\infty$ is closable, (Fukushima et al., [5, Condition 1.1.3, p. 4]).

(3) Let $(\mathbb{D}_1, \mathcal{E}_1)$ be the smallest closed extension of $(\mathcal{D}A \cap L^\infty, \mathcal{E}_1)$. Let be $u \in \mathbb{D}$ and $u_n \in \mathcal{D}A \cap L^\infty$, with $u_n \to u$ in $\mathbb{D}$. Using $\mathcal{E}_1[u_n - u_m] \leqslant \|f\|_\infty \mathcal{E}[u_n - u_m]$ and the closedness of $\mathcal{E}_1$ we get $u_n \to u$ in $\mathbb{D}_1$, hence $\mathbb{D} \subset \mathbb{D}_1$. Now by usual inequalities we see that $\Gamma[u_n]$ is a Cauchy sequence in $L^1(\mathbb{P}_1)$ and that the limit $\Gamma_1[u]$ does not depend on the particular sequence (u_n) satisfying the above condition. Then following Bouleau [2, Chapter III, Section 2.5, p. 38], the functional calculus extends to $\mathbb{D}_1$, the axioms of error structures are fulfilled for $(\Omega, \mathcal{A}, \mathbb{P}_1, \mathbb{D}_1, \Gamma_1)$ and this gives with usual arguments the (b) of the statement. □

1.3. Rajchman measures

In the whole paper, if x is a real number, $[x]$ denotes the entire part of x and $\{x\} = x - [x]$ the fractional part.

Definition 1. A measure μ on the torus $\mathbb{T}^1$ is said to be Rajchman if

$$\hat{\mu} = \int_{\mathbb{T}^1} e^{2i\pi nx}\, d\mu(x) \to 0 \quad \text{when } |n| \uparrow \infty.$$

The set of Rajchman measures $\mathcal{R}$ is a band: if $\mu \in \mathcal{R}$ and if $\nu \ll |\mu|$ then $\nu \in \mathcal{R}$, cf. Rajchman [18,19], Lyons [15].

Lemma. *Let X be a real random variable and let $\Psi_X(u) = \mathbb{E}e^{iuX}$ be its characteristic function. Then*

$$\lim_{|u|\to\infty} \Psi_X(u) = 0 \quad \Leftrightarrow \quad \mathbb{P}_{\{X\}} \in \mathcal{R}.$$

Proof. (a) If $\lim_{|u|\to\infty} \Psi_X(u) = 0$ then $\Psi_X(2\pi n) = (\mathbb{P}_{\{X\}})\widehat{\ }(n) \to 0$.

(b) Let ρ be a probability measure on $\mathbb{T}^1$ such that $\rho \in \mathcal{R}$. From

$$e^{2i\pi ux} = e^{2i\pi[u]x} \sum_{p=0}^{\infty} \frac{((u-[u])2i\pi x)^p}{p!}$$

we have

$$\int e^{2i\pi ux}\rho(dx)=\sum_{p=0}^{\infty}\frac{((u-[u])2i\pi)^p}{p!}a_p\big([u]\big)$$

with $a_p(n)=\int x^p e^{2i\pi nx}\rho(dx)$ hence $|a_p(n)|\leqslant 1$ and $\lim_{|n|\to\infty}a_p(n)=0$. Since $x^p\rho(dx)\in\mathcal{R}$, so

$$\lim_{|u|\to\infty}\int e^{2i\pi ux}\rho(dx)=0.$$

Now if $\mathbb{P}_{\{X\}}\in\mathcal{R}$, since $1_{\{x\in[p,p+1[\}}.\mathbb{P}_{\{X\}}\ll\mathbb{P}_{\{X\}}$ we have

$$\lim_{|u|\to\infty}\mathbb{E}\big[e^{2i\pi uX}\big]=\lim_{|u|\to\infty}\sum_p\mathbb{E}\big[e^{2i\pi uX}1_{\{X\in[p,p+1[\}}\big]$$

which goes to zero by dominated convergence. □

A probability measure on $\mathbb{R}$ satisfying the conditions of the lemma will be called Rajchman.

Examples. Thanks to the Riemann–Lebesgue lemma, absolutely continuous measures are in $\mathcal{R}$. It follows from the lemma that if a measure ν satisfies $\nu\star\cdots\star\nu\in\mathcal{R}$ then $\nu\in\mathcal{R}$. There are singular Rajchman measures, cf. Kahane and Salem [13].

The preceding definitions and properties extend to $\mathbb{T}^d$: a measure μ on $\mathbb{T}^d$ is said to be in $\mathcal{R}$ if $\hat{\mu}(k)\to 0$ as $k\to\infty$ in $\mathbb{Z}^d$. The set of measures in $\mathcal{R}$ is a band. If X is $\mathbb{R}^d$-valued, $\lim_{|u|\to\infty}\mathbb{E}e^{i\langle u,X\rangle}=0$ is equivalent to $\mathbb{P}_{\{X\}}\in\mathcal{R}$ where $\{x\}=(\{x_1\},\dots,\{x_d\})$.

2. Finite-dimensional cases

In the whole article $\stackrel{d}{\Longrightarrow}$ denotes the convergence in law, i.e. the convergence of the probability laws on bounded continuous functions. The *arbitrary functions principle* may be stated as follows.

Proposition 1. *Let X, Y, Z be random variables with values in $\mathbb{R}$, $\mathbb{R}$, and $\mathbb{R}^m$, respectively. Then*

$$\big(\{nX+Y\},X,Y,Z\big)\stackrel{d}{\Longrightarrow}(U,X,Y,Z),\tag{7}$$

where U is uniform on the unit interval independent of (X,Y,Z), if and only if $\mathbb{P}_X$ is Rajchman.

Proof. If μ is a probability measure on $\mathbb{T}^1\times\mathbb{R}^m$, let us put

$$\hat{\mu}(k,\zeta)=\int e^{2i\pi kx+\langle\zeta,y\rangle}\mu(dx,dy).$$

Then $\mu_n\stackrel{d}{\Longrightarrow}\mu$ iff $\hat{\mu}_n(k,\zeta)\to\hat{\mu}(k,\zeta)$ $\forall k\in\mathbb{Z}$, $\forall\zeta\in\mathbb{R}^m$.

(a) If $\mathbb{P}_X \in \mathcal{R}$,

$$\hat{\mathbb{P}}_{(\{nX+Y\},X,Y,Z)}(k,\zeta_1,\zeta_2,\zeta_3) = \mathbb{E}\big[\exp\big\{2i\pi k(nX+Y) + i\zeta_1 X + i\zeta_2 Y + i\langle\zeta_3, Z\rangle\big\}\big]$$
$$= \int e^{2i\pi knx} f(x)\mathbb{P}_{\{X\}}(dx)$$

with $f(x) = \mathbb{E}[\exp\{2i\pi kY + i\zeta_1 X + i\zeta_2 Y + i\langle\zeta_3, Z\rangle\}|\{X\} = x]$. The fact that $f.\mathbb{P}_{\{X\}} \in \mathcal{R}$ gives the result.

(b) Conversely, taking $(k,\zeta_1,\zeta_2,\zeta_3) = (1,0,-2\pi,0)$ gives $\widehat{\mathbb{P}}_{\{X\}}(n) \to 0$ i.e. $\mathbb{P}_X \in \mathcal{R}$. □

Let us suppose now that Y is an $\mathbb{R}^d$-valued random variable, measured with an equidistant graduation corresponding to an orthonormal rectilinear coordinate system, and estimated to the nearest graduation component by component. Thus we put

$$Y_n = Y + \frac{1}{n}\theta(nY)$$

with $\theta(y) = (\frac{1}{2} - \{y_1\}, \ldots, \frac{1}{2} - \{y_d\})$. Let us emphasize that Y_n is a deterministic function of Y.

Theorem 4.

(a) *If $\mathbb{P}_Y$ is Rajchman and if X is $\mathbb{R}^m$-valued*

$$\big(X, n(Y_n - Y)\big) \overset{d}{\Longrightarrow} \big(X, (V_1, \ldots, V_d)\big) \tag{8}$$

where the V_is are independent identically and uniformly distributed on $(-\frac{1}{2}, \frac{1}{2})$ and independent of X.

For all $\varphi \in \mathcal{C}^1 \cap \mathrm{lip}(\mathbb{R}^d)$

$$\big(X, n\big(\varphi(Y_n) - \varphi(Y)\big)\big) \overset{d}{\Longrightarrow} \left(X, \sum_{i=1}^d V_i \varphi_i'(Y)\right), \tag{9}$$

$$n^2\mathbb{E}\big[\big(\varphi(Y_n) - \varphi(Y)\big)^2 \bigm| Y = y\big] \to \frac{1}{12}\sum_{i=1}^d \varphi_i'^2(y) \quad \text{in } L^1(\mathbb{P}_Y). \tag{10}$$

In particular

$$n^2\mathbb{E}\big[\big(\varphi(Y_n) - \varphi(Y)\big)^2\big] \to \mathbb{E}_Y\left[\frac{1}{12}\sum_{i=1}^d \varphi_i'^2(y)\right]. \tag{11}$$

(b) *If φ is of class $\mathcal{C}^2$, the conditional expectation $n^2\mathbb{E}[\varphi(Y_n) - \varphi(Y)|Y = y]$ possesses a version $n^2(\varphi(y + \frac{1}{n}\theta(ny)) - \varphi(y))$ independent of the probability measure $\mathbb{P}$ which converges in the sense of distributions to the function $\frac{1}{24}\Delta\varphi$.*

(c) *If $\mathbb{P}_Y \ll dy$ on $\mathbb{R}^d$, $\forall\psi \in L^1([0,1])$*

$$\big(X, \psi\big(n(Y_n - Y)\big)\big) \overset{d}{\Longrightarrow} \big(X, \psi(V)\big). \tag{12}$$

(d) *We consider the bias operators on the algebra $\mathcal{C}_b^2$ of bounded functions with bounded derivatives up to order 2 with the sequence $\alpha_n = n^2$. If $\mathbb{P}_Y \in \mathcal{R}$ and if one of the following condition is fulfilled*:
 (i) $\forall i = 1, \ldots, d$ *the partial derivative $\partial_i \mathbb{P}_Y$ in the sense of distributions is a measure $\ll \mathbb{P}_Y$ of the form $\rho_i \mathbb{P}_Y$ with $\rho_i \in L^2(\mathbb{P}_Y)$,*
 (ii) $\mathbb{P}_Y = h 1_G \frac{dy}{|G|}$ *with G open set, $h \in H^1 \cap L^\infty(G)$, $h > 0$,*
 then hypotheses (H1)–(H4) *are satisfied and*

$$\begin{aligned}
\overline{A}[\varphi] &= \frac{1}{24}\Delta\varphi, \\
\widetilde{A}[\varphi] &= \frac{1}{24}\Delta\varphi + \frac{1}{24}\sum \varphi_i' \rho_i \quad \big(\textit{case } (\mathrm{i})\big), \\
\widetilde{A}[\varphi] &= \frac{1}{24}\Delta\varphi + \frac{1}{24}\frac{1}{h}\sum h_i'\varphi_i' \quad \big(\textit{case } (\mathrm{ii})\big), \\
\Gamma[\varphi] &= \frac{1}{12}\sum \varphi_i'^2.
\end{aligned}$$

Proof. The argument for relation (8) is similar to the one-dimensional case stated in Proposition 1. The relation (9) comes from the Taylor expansion

$$\varphi(Y_n) - \varphi(Y) = \sum_{i=1}^d (Y_{n,i} - Y_i) \int_0^1 \varphi_i'\big(Y_{n,1}, \ldots, Y_{n,i-1}, Y_i + t(Y_{n,i} - Y_i), Y_{i+1}, \ldots, Y_d\big)\, dt$$

and the convergence

$$\left(X, \sum_i \theta(nY_i)\varphi_i'(Y)\right) \stackrel{d}{\Longrightarrow} \left(X, \sum_i \varphi_i'(Y) V_i\right),$$

thanks to (8) and the following approximation in L^1:

$$\mathbb{E}\left|\sum_i \theta(nY_i)\varphi_i'(Y) - \sum_i \theta(nY_i) \int_0^1 \varphi_i'\big(\ldots, Y_i + t(Y_{n,i} - Y_i), \ldots\big)\, dt\right| \to 0.$$

To prove the formulas (10) and (11) let us remark that

$$\begin{aligned}
&n^2 \mathbb{E}\big[\big(\varphi(Y_n) - \varphi(Y)^2\big) \mid Y = y\big] \\
&\quad = \mathbb{E}\left[\left|\sum_i \theta(nY_i) \int_0^1 \varphi_i'\big(\ldots, Y_i + t(Y_{n,i} - Y_i), \ldots\big)\, dt\right|^2 \,\middle|\, Y = y\right] \\
&\quad = \left|\sum_i \theta(ny_i) \int_0^1 \varphi_i'\left(y_1 + \frac{1}{n}\theta(ny_1), \ldots, y_i + t\frac{1}{n}\theta(ny_i), \ldots\right) dt\right|^2 \quad \mathbb{P}_Y\text{-a.s.}
\end{aligned}$$

Each term $(\theta(ny_i)\int_0^1 \varphi_i'(\ldots)dt)^2$ converges to $\int \theta^2 \varphi_i'^2(y) = \frac{1}{12}\varphi_i'^2$ in L^1 and each term $\theta(ny_i)\theta(ny_j)\int_0^1 \ldots \int_0^1 \ldots$ goes to zero in L^1 what proves the part (a) of the statement.

The part (b) is obtained following the same lines with a Taylor expansion up to second order and an integration by part thanks to the fact that φ is now supposed to be $\mathcal{C}^2$.

In order to prove (c) let us suppose first that $\mathbb{P}_Y = 1_{[0,1]^d}.dy$. Considering a sequence of functions $\psi_k \in \mathcal{C}_b$ tending to ψ in L^1. We have the bound

$$\begin{aligned}
&\left|\mathbb{E}\left[e^{i\langle u,X\rangle} e^{iv\psi(\theta(nY))}\right] - \mathbb{E}\left[e^{i\langle u,X\rangle} e^{iv\psi_k(\theta(nY))}\right]\right| \\
&\quad \leqslant |v| \int \left|\psi\big(\theta(ny)\big) - \psi_k\big(\theta(ny)\big)\right| dy \\
&\quad = |v| \sum_{p_1=0}^{n-1} \cdots \int_{p_1}^{p_1+1} \cdots \left|\psi\big(\theta(ny_1), \ldots\big) - \psi_k(\theta(ny_1), \ldots)\right| dy_1 \ldots dy_d \\
&\quad = |v| \sum \cdots \sum \int \cdots \int \left|\psi\big(\theta(x_1), \ldots\big) - \psi_k\big(\theta(x_1), \ldots\big)\right| \frac{dx_1}{n} \cdots \frac{dx_d}{n} \\
&\quad = |v| \|\psi - \psi_k\|_{L^1}.
\end{aligned}$$

This yields (12) in this case. Now if $\mathbb{P}_Y \ll dy$ then $\mathbb{P}_{\{Y\}} \ll dy$ on $[0,1]^d$ and the weak convergence under dy on $[0,1]^d$ implies the weak convergence under $\mathbb{P}_{\{Y\}}$ what yields the result.

In (d) the point (i) is proved by the approach already used in Bouleau [3] consisting of proving that hypothesis (H3) is fulfilled by displaying the operator $\widetilde{A}$ thanks to an integration by parts. The point (ii) is an application of the Girsanov-type Theorem 3. □

Remarks. (1) About the relations (9)–(11), let us note that with respect to the form

$$\mathcal{E}[\varphi] = \frac{1}{24}\mathbb{E}_Y \sum_i \varphi_i'^2$$

when it is closable, the random variable $\sum_i V_i \varphi_i'$ appears to be *a gradient*: if we put $\varphi^{\#} = \sum_i V_i \varphi_i'$ then we have

$$\mathbb{E}\left[\varphi^{\#2}\right] = \frac{1}{12}\sum_i \varphi_i'^2 = \Gamma[\varphi],$$

the square field operator associated to $\mathcal{E}$. We will find this phenomenon again on the Wiener space.

(2) *Approximation to the nearest graduation, by excess, or by default*. When the approximation is done to the nearest graduation, on the algebra $\mathcal{C}_b^2$ the four bias operators are obtained in Theorem 4 with the sequence $\alpha_n = n^2$ (with $\alpha_n = n$ the four bias operators would be zero).

We would obtain a quite different result with an approximation by default or by excess because of the dominating effect of the shift.

If the random variable Y is approximated by default by $Y_n^{(d)} = \frac{[nY]}{n}$ then

$$n\big(Y_n^{(d)} - Y\big) \overset{d}{\Longrightarrow} -U \quad \text{and} \quad \mathbb{E}\big[n\big(Y_n^{(d)} - Y\big)\big] \to -\frac{1}{2}$$

as soon as Y is say bounded. With this approximation, if we do not erase the shift down proportional to $-\frac{1}{2n}$, and if we take $\alpha_n = n$ we obtain first-order bias operators without diffusion: $\overline{A}[\varphi] = -\frac{1}{2}\varphi' = -\underline{A}[\varphi]$ and $\widetilde{A} = 0$. The same happens of course with the approximation by excess.

(3) *Extension to more general graduations.* Let Y be an $\mathbb{R}^d$-valued random variable approximated by $Y_n = Y + \xi_n(Y)$ with a sequence $\alpha_n \uparrow \infty$ on the algebra $\mathcal{D} = \mathcal{L}\{e^{\langle u,x\rangle},\ u \in \mathbb{R}^d\}$, the function ξ_n satisfying:

$$\begin{cases} \alpha_n \mathbb{E}[|\xi_n|^3(Y)] \to 0; \\ \alpha_n \mathbb{E}[\varphi(Y)\langle u, \xi_n(Y)\rangle^2] \to \mathbb{E}_Y[\varphi.u^* \underline{\underline{\gamma}} u] \quad \forall \varphi \in \mathcal{D}, \forall u \in \mathbb{R}^d, \\ \quad \text{with } \gamma_{ij} \in L^\infty(\mathbb{P}_Y) \text{ and } \frac{\partial \gamma_{ij}}{\partial x_j} \text{ in distributions sense } \in L^2(\mathbb{P}_Y); \\ \alpha_n \mathbb{E}[\varphi(Y)\langle u, \xi_n(Y)\rangle] \to 0 \quad \forall \varphi \in \mathcal{D}. \end{cases} \tag{$*$}$$

Under these hypotheses we have

Theorem 4bis.

(a) (H1) *is satisfied and* $\overline{A}[\varphi] = \frac{1}{2}\sum_{ij} \gamma_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$.

(b) *If for* $i = 1, \dots, d$, *the partial derivative* $\partial_i \mathbb{P}_Y$ *in the sense of distributions is a bounded measure of the form* $\rho_i \mathbb{P}_Y$ *with* $\rho_i \in L^2(\mathbb{P}_Y)$ *then assumptions* (H1)–(H4) *are fulfilled and* $\forall \varphi \in \mathcal{D}$

$$\widetilde{A}[\varphi] = \frac{1}{2}\sum_{ij} \gamma_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_i \left(\sum_j \left(\frac{\partial \gamma_{ij}}{\partial x_j} + \gamma_{ij}\rho_j \right) \right) \frac{\partial \varphi}{\partial x_i}$$

the square field operator is $\Gamma[\varphi] = \sum_{ij} \gamma_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j}$.

Proof. The argument is simple thanks to the choice of the algebra $\mathcal{D}$ and consists of elementary Taylor expansions to prove the existence of the bias operators. Then Theorem 1 applies. □

Historical comment. In its intuitive version, the idea underlying the arbitrary functions method is ancient. The historian J. von Plato [16] dates it back to a book of J. von Kries [12]. We find indeed in this philosophical treatise the idea that if a roulette had equal and infinitely small black and white cases, then there would be an equal probability to fall on a case or on the neighbour one, hence by addition an equal probability to fall either on black or on white. But no precise proof was given. The idea remains at the common sense level.

A mathematical argument for the fairness of the roulette and for the equi-distribution of other mechanical systems (little planets on the Zodiac) was proposed by H. Poincaré in his course on probability published in 1912 [17, Chapter VIII, §92, and especially §93]. In present language, Poincaré shows the weak convergence of $tX + Y \bmod 2\pi$ when $t \uparrow \infty$ to the uniform law on

$(0, 2\pi)$ when the pair (X, Y) has a density. He uses the characteristic functions. His proof supposes the density be $\mathcal{C}^1$ with bounded derivative in order to perform an integration by parts, but the proof would extend to the general case if we were using instead the Riemann–Lebesgue lemma.

The question is then developed without major changes by several authors, E. Borel [1] (case of continuous density), M. Fréchet [4] (case of Riemann-integrable density), B. Hostinský [9,10] (bidimensional case) and is tackled anew by E. Hopf [6–8] with the more general point of view of asymptotic behaviour of dissipative dynamical systems. Hopf has shown that these phenomena are related to mixing and belong to the framework of ergodic theory.

3. Infinite-dimensional extensions of the arbitrary functions principle

3.1. Rajchman-type martingales

Let $(\mathcal{F}_t)$ be a right continuous filtration on $(\Omega, \mathcal{A}, \mathbb{P})$ and M be a continuous local $(\mathcal{F}_t, \mathbb{P})$-martingale nought at zero. M will be said to be Rajchman if the measure $d\langle M, M\rangle_s$ restricted to compact intervals belongs to $\mathcal{R}$ almost surely. We will show that the method followed by Rootzén [21] extends to Rajchman martingales and provides the following theorem.

Theorem 5. *Let M be a continuous local martingale which is Rajchman and such that $\langle M, M\rangle_\infty = \infty$. Let f be a bounded Riemann-integrable periodic function with unit period on $\mathbb{R}$ such that $\int_0^1 f(s)\,ds = 0$. Then for any random variable X*

$$\left(X, \int_0^{\cdot} f(ns)\,dM_s\right) \xrightarrow{d} (X, W_{\|f\|^2\langle M,M\rangle_\cdot}). \tag{13}$$

The weak convergence is understood on $\mathbb{R} \times \mathcal{C}([0,1])$ and W is an independent standard Brownian motion.

Before proving the theorem, let us remark that it shows that the random measure dM_s behaves in some sense like a Rajchman measure. Indeed if $\mathbb{P}_Y \in \mathcal{R}$ we have

$$\int_{-\infty}^{y} g(nx)\mathbb{P}_Y(dx) \to \int_0^1 g(x)\,dx \int_{-\infty}^{y} \mathbb{P}_Y(dx)$$

as soon as g is periodic with unit period, Riemann-integrable and bounded. Now applying the theorem to the Brownian motion gives the similar relation

$$\int_0^t f(ns)\,dB_s \xrightarrow{d} \left(\int_0^1 f^2(s)\,ds\right)^{1/2} \int_0^t dW_s.$$

Proof. We consider the local martingale $N_t = \int_0^t f(ns)\,dM_s$.

(a) In order to be sure that $\langle N, N\rangle_\infty = \infty$, we change N_t into $\widetilde{N}_t = \int_0^t f_n(s)\,dM_s$ with $f_n(s) = f(ns)$ for $s \in [0,1)$, $f_n(s) = 0$ for $s \in [1,n]$ and $f_n(s) = 1$ for $t > n$. We put $S_n(t) = \inf\{s\colon \langle \widetilde{N}, \widetilde{N}\rangle_s > t\}$.

(b) We want to show

$$\mathbb{E}\big[\xi F(\widetilde{N}_{S_n})\big] \to \mathbb{E}\big[\xi F(W)\big] \quad \forall \xi \in L^1(\mathbb{P})\ \forall F \in \mathcal{C}_b\big([0,1]\big). \tag{14}$$

It is enough to consider the case $\xi > 0$, $\mathbb{E}\xi = 1$, and ξ may be supposed to be $\mathcal{F}_T$-measurable for a deterministic time T large enough. Let be $\widetilde{\mathbb{P}} = \xi.\mathbb{P}$ and $D(t) = \mathbb{E}[\xi \mid \mathcal{F}_t]$. The process

$$\widetilde{M}_t = M_t - \int_0^t D^{-1}(s)\,d\big\langle M, D^c\big\rangle_s$$

is a continuous local martingale under $\widetilde{\mathbb{P}}$. Therefore $\int_0^{S_n(t)} f_n(s)\,d\widetilde{M}_s$ is a Brownian motion under $\widetilde{\mathbb{P}}$ (Revuz and Yor [20, Theorem 1.4, pp. 313, 173]). Writing

$$\int_0^{S_n(t)} f_n(s)\,dM_s = \int_0^{S_n(t)} f_n(s)\,d\widetilde{M}_s + \int_0^{S_n(t)} \frac{f_n(s)}{D(s)}\,d\big\langle M, D^c\big\rangle_s$$

and noting that $d\langle M, D^c\rangle_s$ vanishes on $]T, \infty[$, in order to show (14) it suffices to show

$$\sup_{0\leqslant t\leqslant T}\left|\int_0^t \frac{f_n(s)}{D(s)}\,d\big\langle M, D^c\big\rangle_s\right| \to 0 \quad \text{a.s. when } n \to \infty,$$

hence to show

$$\sup_{0\leqslant t\leqslant 1}\left|\int_0^t \frac{f(ns)}{D(s)}\,d\big\langle M, D^c\big\rangle_s\right| \to 0 \quad \text{a.s. when } n \to \infty$$

and, because M is Rajchman this comes from the following lemma.

Lemma. *Let f be as in the statement of the theorem, then $\forall \mu \in \mathcal{R}$*

$$\sup_{0\leqslant t\leqslant 1}\left|\int_0^t f(ns)\mu(ds)\right| \to 0 \quad \textit{as } n \to \infty.$$

Proof. We have

$$\int_0^t f(ns)\mu(ds) \to \int_0^1 f(s)ds \int_0^t \mu(ds) = 0.$$

Since f is bounded, the functions $\int_0^t f(ns)\mu(ds)$ are equi-continuous and the result follows from Ascoli theorem. $\square$

(c) This proves the following stable convergence:

$$\left(X, \int_0^{T_n(.)} f(ns)\,dM_s\right) \stackrel{d}{\Longrightarrow} (X, W_.)$$

and by the fact that the limit

$$\int_0^t f^2(ns)\,d\langle M, M\rangle_s \to \int_0^1 f^2(s)\,ds\langle M, M\rangle_t$$

is a continuous process, this gives the announced result. $\square$

Remark. If $\int_0^1 f(s)\,ds \neq 0$, then keeping the other hypotheses unchanged, we obtain

$$\left(X, \int_0^{.} f(ns)\,dM_s\right) \stackrel{d}{\Longrightarrow} \left(X, \left(\int_0^1 f(s)\,ds\right)M_. + \left(\int_0^1 \left(f - \int_0^1 f\right)^2\right)^{1/2} W_{\langle M,M\rangle_.}\right).$$

We study now the induced limit quadratic form when the martingale M is approximated by the martingale $M_t^n = M_t + \int_0^t \frac{1}{n} f(ns)\,dM_s$. The notation is the same as in the preceding section and f satisfies the same hypotheses as in Theorem 5.

Theorem 6. *Let M be a Rajchman martingale such that $M_1 \in L^2$ and η, ζ bounded adapted processes. Then*

$$n^2\mathbb{E}\left[\left(\exp\left\{i\int_0^1 \eta_s\,dM_s^n\right\} - \exp\left\{i\int_0^1 \eta_s\,dM_s\right\}\right)\left(\exp\left\{i\int_0^1 \zeta_s\,dM_s^n\right\} - \exp\left\{i\int_0^1 \zeta_s\,dM_s\right\}\right)\right]$$
$$\to -\mathbb{E}\left[\exp\left\{i\int_0^1 (\eta_s + \zeta_s)\,dM_s\right\}\int_0^1 \eta_s\zeta_s\,d\langle M, M\rangle_s\right]\int_0^1 f^2(s)\,ds.$$

Proof. By the fundamental formula of calculus (finite increments formula), the first term in the statement may be written

$$-\mathbb{E}\left[\exp\left\{i\int_0^1 (\eta_s + \zeta_s)\,dM_s\right\}\int_0^1 \eta_s f(ns)\,dM_s \int_0^1 \zeta_s f(ns)\,dM_s\right] + o(1).$$

Therefore, thanks to Theorem 5, the statement is a consequence of the following lemma.

Lemma. *Suppose $\mathbb{E}M_1^2 < \infty$ and η adapted and bounded, then the random variables $\int_0^1 \eta_s f(ns)\, dM_s$ are uniformly integrable.*

Proof. It suffices to remark that their L^2-norm is equal to $\mathbb{E}\int_0^1 \eta_s^2 f^2(ns)\, d\langle M, M\rangle_s$ hence uniformly bounded. $\Box$

3.2. Sufficient closability conditions on the Wiener space

The closability problem of the limit quadratic forms obtained in the preceding section may be tackled with the tools available on the Wiener space.

Let us approximate the Brownian motion $(B_t)_{t\in[0,1]}$ by the process $B_t^n = B_t + \int_0^t \frac{1}{n} f(ns)\, dB_s$ where f satisfies the same hypotheses as before.

Theorem 7.

(a) *Let $\xi \in L^2([0,1])$, and let X be a random variable defined on the Wiener space, i.e. a Wiener functional. Then*

$$\left(X, n\left(\exp\left\{i\int_0^1 \xi\, dB^n\right\} - \exp\left\{i\int_0^1 \xi\, dB\right\}\right)\right) \overset{d}{\Longrightarrow} \left(X, \|f\|_{L^2}\left(\exp\left\{i\int_0^1 \xi\, dB\right\}\right)^{\#}\right), \tag{15}$$

here for any regular Wiener functional Z we put $Z^{\#}(\omega, w) = \int_0^1 D_s Z\, dW_s$, where W is an independent Brownian motion.

(b) $$n^2\mathbb{E}\left[\left(e^{i\xi.B^n} - e^{i\xi.B}\right)^2\right] \to -\mathbb{E}\left[e^{2i\xi.B}\right]\int_0^1 \xi^2\, ds \|f\|_{L^2}^2 \tag{16}$$

on the algebra $\mathcal{L}\{e^{i\xi.B}\}$ the quadratic form $-\frac{1}{2}\mathbb{E}[e^{2i\xi.B}]\int_0^1 \xi^2\, ds$ is closable, its closure is the Ornstein–Uhlenbeck form.

Proof. (a) The first assertion comes easily from the similar result concerning Rajchman martingales using the fact that $\int_0^1 e^{i\alpha\int_0^1 \frac{1}{n} f(ns)\, dB_s}\, d\alpha \to 1$ in L^p $p \in [1, \infty[$.

(b) The obtained quadratic form is immediately recognized as the Ornstein–Uhlenbeck form which is closed. It follows that hypothesis (H3) is fulfilled and the symmetric bias operator is

$$\widetilde{A}\left[e^{i\int \xi\, dB}\right] = \left(-\frac{i}{2}\int \xi\, dB - \frac{1}{2}\int \xi^2\, ds\right)e^{i\int \xi\, dB}. \qquad \Box$$

If instead of the Wiener measure m we consider the measure $m_1 = h.m$ for an $h > 0$, $h \in \mathbb{D}_{\mathrm{ou}} \cap L^\infty$ where $\mathbb{D}_{\mathrm{ou}}$ $(= D^{2,1})$ denotes the domain of the Ornstein–Uhlenbeck form, we know by the Girsanov-type Theorem 3 that the form

$$-\frac{1}{2}\mathbb{E}_1\left[e^{2i\xi.B}\int_0^1 \xi^2\, ds\right]$$

is closable, admits the same square field operator on $\mathbb{D}_{\text{ou}}$, and that its generator A_1 satisfies

$$A_1[\varphi] = \widetilde{A}[\varphi] + \frac{1}{2h}\Gamma_{\text{ou}}[\varphi, h] \quad \text{for } \varphi \in \mathcal{D}A_{\text{ou}}.$$

Since the point (a) of the theorem is still valid under m_1 because of the properties of stable convergence, the preceding theorem is still valid under m_1, the Dirichlet form being now

$$\mathcal{E}_1[\varphi] = \frac{1}{2}\mathbb{E}_1\big[\Gamma_{\text{ou}}[\varphi]\big] \quad \text{for } \varphi \in \mathbb{D}_{\text{ou}}.$$

Remark. Let us come back to the general case of Rajchman martingales. If we suppose the Rajchman local martingale M is in addition Gaussian, which is equivalent to suppose $\langle M, M\rangle$ deterministic, then on the algebra $\mathcal{L}\{e^{i\int \xi\, dM};\ \xi$ deterministic bounded$\}$ the limit quadratic form

$$-\mathbb{E}\left[e^{i\int(\eta+\zeta)\,dM}\int_0^1 \zeta_s\eta_s\, d\langle M, M\rangle_s\right]\|f\|_{L^2}^2$$

is closable, hence (H3) is satisfied.

Indeed, it suffices to exhibit the corresponding symmetric bias operator. But by the use of the calculus for Gaussian variables, it is easily seen that the operator defined by

$$\widetilde{A}\big[e^{i\int \xi\, dM}\big] = e^{i\int \xi dM}\left(-\frac{i}{2}\int \xi\, dM - \frac{1}{2}\int \xi^2\, d\langle M, M\rangle_s\right)\int f^2\, ds$$

satisfies the required condition. □

3.3. Approximation of the Ornstein–Uhlenbeck gradient

Let m be the Wiener measure on $\mathcal{C}([0,1],\mathbb{R})$. Let θ be a real periodic function of period 1 such that $\int_0^1 \theta(s)\, ds = 0$ and $\int_0^1 \theta^2(s)\, ds = 1$. We consider the transformation R_n of the space $L^2_{\mathbb{C}}(m)$ defined by its action on the chaos: if $X = \int_{s_1<\cdots<s_k} \hat{f}(s_1,\dots,s_k)\, dB_{s_1}\dots dB_{s_k}$ for $\hat{f} \in L^2_{\text{sym}}([0,1]^k,\mathbb{C})$, then

$$R_n(X) = \int_{s_1<\cdots<s_k} \hat{f}(s_1,\dots,s_k) e^{i\frac{1}{n}\theta(ns_1)}\, dB_{s_1}\dots e^{i\frac{1}{n}\theta(ns_k)}\, dB_{s_k}.$$

Since $\|X\|^2_{L^2(m)} = \int_{s_1<\cdots<s_k} |\hat{f}|^2\, ds_1\dots ds_k = \frac{1}{k!}\|\hat{f}\|^2_{L^2_{\text{sym}}}$, R_n is an isometry from $L^2_{\mathbb{C}}(m)$ into itself and $\forall \xi \in L^2_{\mathbb{C}}([0,1])$

$$R_n\big[e^{\int \xi\, dB_s - \frac{1}{2}\int \xi^2\, ds}\big] = e^{\int \xi e^{i\frac{1}{n}\theta(ns)}\, dB_s - \frac{1}{2}\int \xi^2 e^{\frac{2i}{n}\theta(ns)}\, ds},$$

$$\big\|e^{\int \xi\, dB_s - \frac{1}{2}\int \xi^2\, ds}\big\|_{L^2_{\mathbb{C}}} = e^{\frac{1}{2}\int |\xi|^2\, ds}.$$

From the relation

$$n\left(e^{\frac{i}{n}\sum_{p=1}^{k}\theta(ns_p)}-1\right)=i\sum_{p=1}^{k}\theta(ns_p)\int_0^1 e^{\alpha\frac{i}{n}\sum_p\theta(ns_p)}\,d\alpha$$

it follows that if X belongs to kth chaos

$$\left\|n\left(R_n(X)-X\right)\right\|_{L^2}^2\leqslant k^2\|X\|^2\|\theta\|_\infty^2$$

then, denoting by A the Ornstein–Uhlenbeck operator, for $X\in\mathcal{D}(A)$ we have

$$\left\|n\left(R_n(X)-X\right)\right\|_{L^2}\leqslant 2\|AX\|\|\theta\|_\infty.$$

So we can state:

Theorem 8. *If $X\in\mathcal{D}(A)$*

$$\left(-in\left(R_n(X)-X\right),B\right)\stackrel{d}{\Longrightarrow}\left(X^\#,B\right)$$

with $X^\#=\int_0^1 D_sX\,dW_s$ where W is an independent Brownian motion.

Proof. If X belongs to the kth chaos, expanding the exponential by its Taylor series gives

$$n\left(R_n(X)-X\right)=i\int\limits_{s_1<\cdots<s_k}\hat{f}(s_1,\ldots,s_k)\sum_{p=1}^{k}\theta(ns_p)\,dB_{s_1}\ldots dB_{s_k}+Q_n$$

with $\|Q_n\|^2\leqslant\frac{1}{4n}k^2\|\theta\|_\infty^2\|X\|^2$.

Now, since $\int_{s_1<\cdots<s_p<\cdots<s_k}h(s_1,\ldots,s_k)\theta(ns_p)\,dB_{s_1}\ldots dB_{s_p}\ldots dB_{s_k}$ converges stably to $\int_{s_1<\cdots<s_p<\cdots<s_k}h(s_1,\ldots,s_k)\,dB_{s_1}\ldots dW_{s_p}\ldots dB_{s_k}$ we obtain that

$$\begin{aligned}
&-in\left(R_n(X)-X\right)\\
&\quad\stackrel{s}{\Longrightarrow}\int\limits_{t<s_2<\cdots<s_k}\hat{f}(t,s_2,\ldots,s_k)\,dW_t\,dB_{s_2}\ldots dB_{s_k}\\
&\qquad+\int\limits_{s_1<t<\cdots<s_k}\hat{f}(s_1,t,\ldots,s_k)\,dB_{s_1}\,dW_t\ldots dB_{s_k}\\
&\qquad\vdots\\
&\qquad+\int\limits_{s_1<\cdots<s_{k-1}<t}\hat{f}(s_1,\ldots,s_{k-1},t)\,dB_{s_1}\ldots dB_{s_{k-1}}\,dW_t
\end{aligned}$$

which is equal to $\int D_s(X)\,dW_s=X^\#$.

For the general case, we approximate X by X_k for the norm $\mathbb{D}^{2,2}$ and reasoning with the characteristic functions yields the result (see the proof of Theorem 10). $\square$

By the properties of the stable convergence, the convergence in law of Theorem 8 still holds under $\widetilde{m}\ll m$.

Theorem 9. $\forall X \in \mathcal{D}(A)$

$$n^2\mathbb{E}\big[\big|R_n(X)-X\big|^2\big] \to 2\mathcal{E}[X],$$

where $\mathcal{E}$ is the Dirichlet form associated with the Ornstein–Uhlenbeck operator.

Proof. As R_n preserves the chaos and the expansion of $n(R_n(X)-X)$ on the chaos is dominated by that of $2\|\theta\|_\infty AX$, it suffices to argue when X is in the kth chaos.

Starting from

$$n^2\mathbb{E}\big|R_n(X)-X\big|^2 = n^2 \int\limits_{s_1<\cdots<s_k} |\hat{f}|^2\big|e^{\frac{i}{n}\sum_p \theta(ns_p)}-1\big|^2\,ds_1\ldots ds_k,$$

expanding the exponential and estimating the remainder we obtain

$$\lim_n n^2\mathbb{E}\big|R_n(X)-X\big|^2 = \frac{k}{k!}\int\limits_{[0,1]^k} |\hat{f}|^2\,ds_1\ldots ds_k \int\limits_0^1 \theta^2\,dt = k\|X\|^2$$

what gives the result. □

Following the same arguments, it is possible to show that the theoretical and practical bias operators $\overline{A}$ and $\underline{A}$ defined on the algebra $\mathcal{L}\{e^{\int \xi\,dB};\ \xi \in \mathcal{C}^1\}$ by

$$n^2\mathbb{E}\big[\big(R_n(X)-X\big)Y\big] = \langle \overline{A}X, Y\rangle_{L^2(m)},$$
$$n^2\mathbb{E}\big[\big(X-R_n(X)\big)R_n(Y)\big] = \langle \underline{A}X, Y\rangle_{L^2(m)}$$

exist and are equal to A.

3.4. Isometries on the Wiener space

Let us now consider a d-dimensional Brownian motion (B_t). Let $t \mapsto M_t$ be a deterministic bounded measurable periodic map with period 1 with values in the space of $d \times d$ orthogonal matrices such that $\int_0^1 M_s\,ds = 0$ (for instance a rotation of angle $2\pi t$). We denote still m the Wiener measure law of (B_t). The transformation $B_t \mapsto \int_0^t M_s\,dB_s$ induces an endomorphism T_M isometric in $L^p(m)$, $1 \leqslant p \leqslant \infty$. We put $M_n(s) = M_{ns}$ and $T_n = T_{M_n}$.

Theorem 10. *Let X be in $L^1(m)$. Let be $\widetilde{m} \ll m$, we have under $\widetilde{m}$:*

$$\big(T_n(X), B\big) \stackrel{d}{\Longrightarrow} \big(X(w), B\big).$$

The convergence in law is understood on $\mathbb{R} \times \mathcal{C}([0,1])$ and $X(w)$ denotes a random variable with the same law as that of X under m, function of a Brownian motion W independent of B.

Proof. (a) If X has the form

$$X = \exp\left\{ i \int_0^1 \xi . dB + \frac{1}{2} \int_0^1 |\xi|^2 \, ds \right\}$$

for an element $\xi \in L^2([0,1], \mathbb{R}^d)$, we have

$$T_n(X) = \exp\left\{ i \int_0^1 \xi_s^* M_n(s) \, dB_s + \frac{1}{2} \int_0^1 |\xi|^2 \, ds \right\},$$

where ξ_s^* denotes the transposed of ξ_s. If we put $Z_t^n = \int_0^t \xi_s^t M_n(s) \, dB_s$ then

$$\langle Z^n, Z^n \rangle_t = \int_0^t \xi_s^t M_n(s) M_n^*(s) \xi_s \, ds = \int_0^t |\xi|^2(s) \, ds$$

is a continuous function. Now by Theorem 4

$$\int_0^t \xi_s^* M_n(s) \, ds \to \int_0^t \xi_s^* \, ds \int_0^1 M_n(s) \, ds = 0.$$

Since the functions $t \mapsto \int_0^t \xi_s^* M_n(s) \, ds$ are uniformly continuous (M is bounded), by Ascoli theorem $\sup_t |\int_0^t \xi_s^* M_n(s) \, ds| \to 0$. The argument of Rootzen applied once more,

$$\left(\int_0^{\cdot} \xi^* M_n \, dB, B \right) \overset{d}{\Longrightarrow} \left(\int_0^{\cdot} \xi . dW, B \right),$$

gives the result by the continuity of the exponential function.

(b) For $X \in L^1(m)$, we consider X_k linear combination of exponentials of the above form approximating X in $L^1(m)$.

By (a) we have $\forall h \in L^2([0,1], \mathbb{R}^d)$

$$\mathbb{E}\left[e^{iuT_n(X_k)} e^{i \int h.dB} \right] \to \mathbb{E}\left[e^{iuX_k} \right] \mathbb{E}\left[e^{i \int h.dB} \right]$$

but

$$\left| \mathbb{E}\left[e^{iuT_n(X)} e^{i \int h.dB} - \mathbb{E}[e^{iuT_n(X_k)} e^{i \int h.dB} \right] \right| \leqslant |u| \mathbb{E}\left| T_n(X) - T_n(X_k) \right| = |u| \|X - X_k\|_{L^1}$$

what gives the result.

(c) This extends to $\widetilde{m} \ll m$ by the properties of stable convergence. $\square$

3.5. Stochastic differential equations from dynamics

In the case $f(x)=\theta(x)=\frac{1}{2}-\{x\}$, the approximation used in Sections 3.1 and 3.2 approaches B_t by

$$B_t-\int_0^t\left(s-\frac{1}{2n}-\frac{[ns]}{n}\right)dB_s \tag{17}$$

and yields the limit of the type

$$\big(n\big(B_.-B^n_.\big),B_.\big)=\left(n\int_0^.\left(s-\frac{1}{2n}-\frac{[ns]}{n}\right)dB_s,B_.\right)\xrightarrow{d}\left(\frac{1}{\sqrt{12}}W_.,B_.\right)$$

and

$$\begin{aligned}&\left(n\int_0^.\left(s-\frac{[ns]}{s}\right)dB_s,n\int_0^.\left(B_s-B_{\frac{[ns]}{n}}\right)ds,B_.\right)\\&\quad\xrightarrow{d}\left(\frac{1}{\sqrt{12}}W_.+\frac{1}{2}B_.,-\frac{1}{\sqrt{12}}W_.+\frac{1}{2}B_.,B_.\right).\end{aligned} \tag{18}$$

Now, when we solve by the Euler method a stochastic differential equation of the type defining a diffusion process and expand the coefficients in series, we encounter integrals of the type

$$\int_0^.\left(s-\frac{[ns]}{n}\right)dB_s,\qquad\int_0^.(B_s-B_{\frac{[ns]}{n}})\,ds$$

but also of the type

$$\int_0^.(B_s-B_{\frac{[ns]}{n}})\,dB_s \tag{19}$$

and these last ones, by the central limit theorem, yield the convergence

$$\left(\sqrt{n}\int_0^.(B_s-B_{\frac{[ns]}{n}})\,dB_s,B_.\right)\xrightarrow{d}\left(\frac{1}{\sqrt{2}}\widetilde{W}_.,B_.\right). \tag{20}$$

Then let us remark that:

(a) the limits (18) are generally hidden by the limits (20) because of the order of magnitude of the coefficients n and $\sqrt{n}$, respectively;
(b) in (20), the conditional law of the random variable $\int_0^. B_{\frac{[ns]}{n}}\,dB_s$ with values in $\mathcal{C}([0,1],\mathbb{R})$ given $\int_0^. B_s\,dB_s$ is not reduced to a Dirac mass, the approximation is not deterministic (as can be seen, for instance, by changing the sign of a Brownian path after the time $T=\inf\{s\rangle\frac{n-1}{n},B_s=0\}$, the σ-field generated by $\int_0^. B_s\,dB_s$ being $\sigma(B_s^2,s\leqslant 1)$). In (17) instead, B^n is a deterministic function of B.

Nevertheless, for some stochastic differential equations the limits (18) remain dominant and determine the convergence. This concerns, for instance, stochastic differential equations of the form

$$\begin{cases} X_t^1 = x_0^1 + \int_0^t f^{11}(X_s^2)\,dB_s + \int_0^t f^{12}(X_s^1, X_s^2)\,ds, \\ X_t^2 = x_0^2 + \int_0^t f^{22}(X_s^1, X_s^2)\,ds, \end{cases} \tag{21}$$

where X^1 has its values in $\mathbb{R}^{k_1}$, X^2 in $\mathbb{R}^{k_2}$, B in $\mathbb{R}^d$ and f^{ij} are matrices with suitable dimensions. Such equations are encountered to describe the movement of mechanical systems under the action of forces with a random noise, when the noisy forces depend on the position of the system and the time. Typically

$$\begin{cases} X_t = X_0 + \int_0^t V_s\,ds, \\ V_t = V_0 + \int_0^t a(X_s, V_s, s)\,ds + \int_0^t b(X_s, s)\,dB_s \end{cases}$$

which is a perturbation of the equation $\frac{d^2x}{dt^2} = a(x, \frac{dx}{dt}, t)$. In such equations the stochastic integral may be understood as Ito as well as Stratonovitch. For Eq. (21) the iterative method of Kurtz and Protter [14] (see also Jacod and Protter [11]) may be applied starting with the results obtained in generalizing of the arbitrary functions principle. This yields the following result that we state in the case $k_1 = k_2 = d = 1$ for simplicity.

Theorem 11. *If functions f^{ij} are $\mathcal{C}_{\mathrm{b}}^1$, and if X^n is the solution of* (21) *by the Euler scheme,*

$$\big(n\big(X^n - X\big), X, B\big) \overset{d}{\Longrightarrow} (U, X, B),$$

where the process U is a solution of the stochastic differential equation

$$U(t) = \sum_{k,j} \int_0^t \frac{\partial f^{ij}}{\partial x_k}(X_s) U_s^k\,dY_s^j - \sum_{k,j} \int_0^t \frac{\partial f^{ij}}{\partial x_k}(X_s) \sum_m f^{km}(X_s)\,dZ_s^{mj},$$

where $Y_s = (B_s, s)^t$ and

$$dZ_s^{12} = \frac{1}{\sqrt{12}}\,dW_s + \frac{1}{2}\,dB_s,$$

$$dZ_s^{21} = -\frac{1}{\sqrt{12}}\,dW_s + \frac{1}{2}\,dB_s,$$

$$dZ_s^{22} = \frac{ds}{2},$$

and as ever W is an independent Brownian motion.

Thus the Euler scheme for solving this kind of equations encountered in mechanics gives rise to an asymptotic weak limit, but in $\frac{1}{n}$ and based on the arbitrary functions principle, instead of being in $\frac{1}{\sqrt{n}}$ and based on a version of the central limit theorem.

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